

COMBINATORICS ON PERMUTATION TABLEAUX OF TYPE A AND TYPE B

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ABSTRACT. We give two bijective proofs of a result of Corteel and Nadeau. We find a generating function related to unrestricted columns of permutation tableaux. As a consequence, we obtain a sign-imbalance formula for permutation tableaux. We extend the first bijection of Corteel and Nadeau between permutations and permutation tableaux to type B objects. Using this type B bijection, we generalize a result of Lam and Williams. We prove that the bijection of Corteel and Nadeau and our type B bijection can be expressed as zigzag maps on the alternative representation.

1. INTRODUCTION

A permutation tableau is a relatively new combinatorial object coming from Postnikov's study of totally nonnegative Grassmanian [10]. As its name suggests, permutation tableaux are in bijection with permutations. Surprisingly, there is also a connection between permutation tableaux and a statistical physics model called partially asymmetric exclusion process (PASEP), see [2, 5, 4]. Recently, several papers on the combinatorics of permutation tableaux have been published, see [1, 3, 9, 15, 17]. In this paper we study in more detail the combinatorics of permutation tableaux of type A and type B.

A *permutation tableau* (of type A) is a Ferrers diagram with possibly empty rows together with a 0, 1-filling of the cells satisfying the following conditions:

- (1) each column has at least one 1,
- (2) there is no 0 which has a 1 above it in the same column and a 1 to the left of it in the same row.

See Figure 2 for an example of a permutation tableau. The *length* of a permutation tableau is defined to be the number of rows plus the number of columns. We denote by $\mathcal{PT}(n)$ the set of permutation tableaux of length n . A 0 of a permutation tableau is called *restricted* if there is a 1 above it in the same column. If a row has no restricted 0, then it is called *unrestricted*.

Let $(x)_n$ denote the rising factorial, i.e. $(x)_0 = 1$ and $(x)_n = x(x+1) \cdots (x+n-1)$ for $n \geq 1$. Note that $(2)_{n-1} = (1)_n = n!$ for $n \geq 1$.

Using recurrence relations, Corteel and Nadeau [3, Proposition 1] proved the following:

$$(1) \quad \sum_{T \in \mathcal{PT}(n)} x^{\text{urr}(T)-1} y^{\text{topone}(T)} = (x+y)_{n-1},$$

where $\text{urr}(T)$ is the number of unrestricted rows of T and $\text{topone}(T)$ is the number of 1's in the first row. In this paper we give two bijective proofs of (1).

Similarly to the definition of unrestricted row, a column of a permutation tableau is called *unrestricted* if it does not have a 0 which has a 1 to the left of it in the same row. For $T \in \mathcal{PT}(n)$, let $\text{urc}(T)$ denote the number of unrestricted columns of T . Then we prove the following.

Theorem 1.1. *We have*

$$\sum_{n \geq 0} \sum_{T \in \mathcal{PT}(n)} t^{\text{urc}(T)} x^n = \frac{1 + E_t(x)}{1 + (t-1)x E_t(x)},$$

where

$$E_t(x) = \sum_{n \geq 1} n(t)_{n-1} x^n.$$

There are two interesting special cases $t = 2$ and $t = -1$ of Theorem 1.1 as follows.

Corollary 1.2. *Let $f(n)$ denote the number of connected permutations of $\{1, 2, \dots, n\}$, see Section 3 for the definition. Then*

$$\sum_{T \in \mathcal{PT}(n)} 2^{\text{urc}(T)} = f(n+1).$$

In Section 3 we also give a combinatorial proof of Corollary 1.2.

We define the *sign* of $T \in \mathcal{PT}(n)$ to be $\text{sign}(T) = (-1)^{\text{urc}(T)}$. When $t = -1$ in Theorem 1.1, we get the following sign-imbalance formula for permutation tableaux.

Corollary 1.3. *We have*

$$\sum_{T \in \mathcal{PT}(n)} \text{sign}(T) = \frac{(1+i)^n + (1-i)^n}{2}.$$

We also consider the set $\mathcal{PT}_B(n)$ of type B permutation tableaux of length n defined by Lam and Williams [8]. Using a similar argument in [3], Lam and Williams [8, Proposition 11.4] showed the following (see Section 4 for the definition of $\text{urr}(T)$ and $\text{diag}(T)$):

$$(2) \quad \sum_{T \in \mathcal{PT}_B(n)} x^{\text{urr}(T)-1} z^{\text{diag}(T)} = (1+z)^n (x+1)_{n-1}.$$

In this paper we find a bijection Φ_B between the type B permutation tableaux and the type B permutations, which extends the first bijection, say Φ , in [3]. Using this bijection we prove the following generalization of (2). See Section 4 for the definition of $\text{urr}(T)$, $\text{diag}(T)$ and $\text{top}_{0,1}(T)$.

Theorem 1.4. *We have*

$$\sum_{T \in \mathcal{PT}_B(n)} x^{\text{urr}(T)-1} y^{\text{top}_{0,1}(T)} z^{\text{diag}(T)} = (1+z)^n (x+y)_{n-1}.$$

There is another representation of a permutation tableau which we call the *alternative representation*. It was first introduced by Viennot [16] and systematically studied by Nadeau [9]. Nadeau [9] found a seemingly different bijection between the permutation tableaux and the permutations using alternative forests. He also showed that despite their different descriptions his bijection is, in fact, the same as the bijection Φ of Corteel and Nadeau [3]. We show that this bijection can also be described as a zigzag map on the alternative representation similar to the zigzag

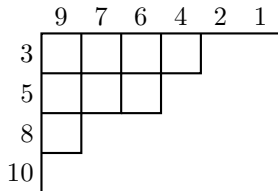


FIGURE 1. A Ferrers diagram with labeled rows and columns.

map on permutation tableaux in [15]. More generally, we show that our map Φ_B can be described as a zigzag on the alternative representation for type B permutation tableaux.

The rest of this paper is organized as follows. In Section 2 we recall the bijection Φ and some properties of it. Then we give two bijective proofs of (1). In Section 3 we prove Theorem 1.1 and Corollaries 1.2 and 1.3. In Section 4 we define the bijection Φ_B and prove Theorem 1.4. In Section 5 we describe Φ and Φ_B in terms of zigzag maps on the alternative representation. In Section 6 we suggest some open problems.

2. TWO BIJECTIVE PROOFS OF CORTEEL AND NADEAU'S THEOREM

A *Ferrers diagram* is a left-justified arrangement of square cells with possibly empty rows and columns. The *length* of a Ferrers diagram is the sum of the number of rows and the number of columns. If a Ferrers diagram is of length n , then we label the steps in the south-east border with $1, 2, \dots, n$ from north-east to south-west. We label a row (resp. column) with i if the row (resp. column) contains the south (resp. west) step labeled with i , see Figure 1. We denote the row (resp. column) labeled with i by Row i (resp. Column i). The (i, j) -*entry* is the cell in Row i and Column j .

For a permutation tableau T , the *alternative representation* of T is the diagram obtained from T by replacing the topmost 1's with \uparrow 's, the rightmost restricted 0's with \leftarrow 's and removing the remaining 0's and 1's, see Figure 2. Here, by a topmost 1 we mean a 1 which is the topmost 1 in its column. A rightmost restricted 0 is similar. Note that we can recover the permutation tableau T from its alternative representation as follows. We fill each empty cell with a 0 if there is an arrow pointing to it and with a 1 otherwise. And then we replace each \uparrow with a 1 and each \leftarrow with a 0. It is easy to see that in the alternative representation of T there is no arrow pointing to another arrow and each column has exactly one \uparrow . Conversely, a filling of a Ferrers diagram with \uparrow 's and \leftarrow 's satisfies these conditions if and only if it is the alternative representation of a permutation tableau.

Remark 2.1. Our definition of the alternative representation comes from alternative tableaux introduced by Viennot [17]. An alternative tableau is obtained by deleting the first row of the alternative representation of a permutation tableau. Alternative tableaux have the nice symmetric property for rows and columns. Rather than using the term alternative tableau, we use the term alternative representation to emphasize that we consider it as a permutation tableau. See [9] for more information on alternative tableaux.

We denote by S_n the set of permutations of $[n] = \{1, 2, \dots, n\}$.

	12	9	8	6	5	3	
1	0	0	1	0	0	1	
2	0	0	0	1	1	1	
4	0	0	0	0	1		
7	0	1	1				
10	1						
11	1						
13							

	12	9	8	6	5	3	
1			↑			↑	
2			←	↑	↑		
4				←			
7		↑					
10	↑						
11							
13							

FIGURE 2. A permutation tableau T (left) and the alternative representation of T (right).

In [3], Corteel and Nadeau found two bijections between permutation tableaux and permutations. We recall their first bijection $\Phi : \mathcal{PT}(n) \rightarrow S_n$. Here we use the alternative representation.

Given $T \in \mathcal{PT}(n)$, we first set π to be the word of labels of the unrestricted rows of T arranged in increasing order. For each column, say Column i , of T , starting from the leftmost column and proceeding to the right, if Row j contains a \uparrow in Column i and $i_1 < \dots < i_r$ are the labels of the rows containing a \leftarrow in Column i , we add i_1, \dots, i_r, i in this (increasing) order before j in π . Then $\Phi(T)$ is the resulting permutation π . For example, if T is the permutation tableau in Figure 2, then

$$(3) \quad \Phi(T) = 4, 6, 5, 2, 8, 3, 1, 9, 7, 12, 10, 11, 13.$$

Corteel and Nadeau [3] proved that $\Phi : \mathcal{PT}(n) \rightarrow S_n$ is a bijection by providing the inverse map.

For now and later use we define the following terminologies.

For a word $w = w_1 \dots w_n$ of distinct integers, w_i is called a *right-to-left maximum*, or a *RL-maximum*, if $w_i > w_j$ for all $j \in \{i+1, i+2, \dots, n\}$. Similarly, w_i is called a *right-to-left minimum*, or a *RL-minimum*, if $w_i < w_j$ for all $j \in \{i+1, i+2, \dots, n\}$. Let w_{i_1}, \dots, w_{i_k} be the RL-maxima of w with $i_1 < i_2 < \dots < i_k$. Note that $i_k = n$. By the *cycles of w according to RL-maxima*, we mean the cycles

$$(w_1, \dots, w_{i_1}), (w_{i_1+1}, \dots, w_{i_2}), \dots, (w_{i_{k-1}+1}, \dots, w_{i_k}).$$

We also define the *cycles of w according to RL-minima* in the same way.

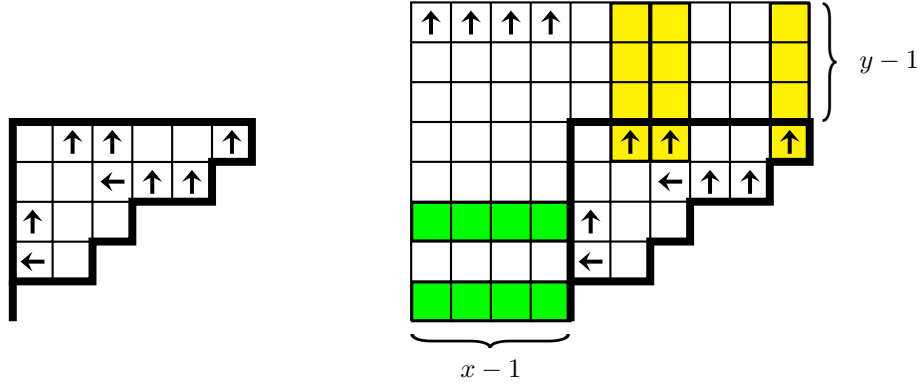
Corteel and Nadeau [3, Theorem 1] proved the first item and Nadeau [9, Theorem 5.4] proved the second item in the following lemma.

Lemma 2.1. *For $T \in \mathcal{PT}(n)$ and $\pi \in S_n$ with $\Phi(T) = \pi$, we have the following.*

- (1) *The labels of the unrestricted rows of T are exactly the RL-minima of π .*
- (2) *The labels of the columns containing a 1 in the first row of T are exactly the RL-maxima of the subword of π which is to the left of the 1 in π .*

For example, the labels of the columns containing a 1 in the first row of T in Figure 2 are 3 and 8 which are the RL-maxima of the subword 4, 6, 5, 2, 8, 3 of $\Phi(T)$ in (3).

Now we give two bijective proofs of the following theorem which is first proved by Corteel and Nadeau [3, Proposition 1] using a recurrence relation. In fact, in

FIGURE 3. The construction of T' from T .

the first proof we give a bijection is between objects counted by the values of the polynomial evaluated at positive integers, whereas in the second proof we give a bijection between objects counted by the coefficients of the polynomials.

Theorem 2.2. *We have*

$$\sum_{T \in \mathcal{PT}(n)} x^{\text{urr}(T)-1} y^{\text{topone}(T)} = (x+y)_{n-1}.$$

First proof of Theorem 2.2. Since both sides of the equation are polynomials in x and y , it is sufficient to prove this for all positive integers x and y . Assume x and y are positive integers and let $N = n + x + y - 2$.

Given $T \in \mathcal{PT}(n)$, we construct $T' \in \mathcal{PT}(N)$ as follows. Firstly, we add $y - 1$ rows and $x - 1$ columns to T by adding $y - 1$ south steps at the beginning and $x - 1$ west steps at the end. Secondly, we fill the first entries of the leftmost $x - 1$ columns with \uparrow 's, see Figure 3.

Let U be the set of labels of the columns of T containing a \uparrow in the first row and let L be the set of labels of the unrestricted rows of T except Row 1. Now consider all permutation tableaux obtained from T' by doing the following: (1) for each $u \in U$, move the \uparrow in the $(y, u + y - 1)$ -entry to the $(i, u + y - 1)$ -entry for some $1 \leq i \leq y$ and (2) for each $\ell \in L$, put a \leftarrow in the $(\ell + y - 1, j)$ -entry for some $N - x + 2 \leq j \leq N$ or do nothing. See Figure 3, where the possible cells for \uparrow 's and \leftarrow 's are colored yellow and green respectively. Clearly, there are $x^{\text{urr}(T)-1} y^{\text{topone}(T)}$ such permutation tableaux for given T . Thus the number of permutation tableaux obtained in this way for all $T \in \mathcal{PT}(n)$ equals the left hand side of the equation in the theorem. Moreover, it is easy to see that such permutation tableaux are exactly those in $\mathcal{PT}(N)$ satisfying the following properties.

- (1) Each integer $1 \leq i \leq y$ is the label of an unrestricted row.
- (2) Each integer $N - x + 2 \leq j \leq N$ is the label of a column containing a \uparrow in the first row.

By Lemma 2.1, $Q \in \mathcal{PT}(N)$ satisfies the above two conditions if and only if in the permutation $\pi = \Phi(Q) \in S_N$ the integers $N, N - 1, \dots, N - x + 2, 1, 2, \dots, y$ are arranged in this order. Since the number of such permutations is $\frac{(n+x+y-2)!}{(x+y-1)!} = (x+y)_{n-1}$, we are done. \square

Second proof of Theorem 2.2. Let $c(n, k)$ denote the number of permutations of $[n]$ with k cycles. The following is well known, see [13, Proposition 1.3.4]:

$$\sum_{k=0}^n c(n, k)x^k = x(x+1)\cdots(x+n-1).$$

Substituting n and x by $n-1$ and $x+y$ respectively in the above equation, we obtain

$$\sum_{i,j} c(n-1, i+j) \binom{i+j}{i} x^i y^j = (x+y)_{n-1}.$$

Thus it is sufficient to show that the number of $T \in \mathcal{PT}(n)$ with $\text{urr}(T) - 1 = i$ and $\text{topone}(T) = j$ is equal to $c(n-1, i+j) \binom{i+j}{i}$. For such a permutation tableau T , let $\pi = \Phi(T)$ and assume that π is decomposed as $\sigma 1 \tau$. By Lemma 2.1, the number of RL-minima of τ is i and the number of RL-maxima of σ is j . Consider the set C_1 of cycles of τ according to RL-minima and the set C_2 of cycles of σ according to RL-maxima. Then C_1 and C_2 have i and j cycles respectively, and their union forms a permutation of $\{2, 3, \dots, n\}$ with $i+j$ cycles. For any permutation of $\{2, 3, \dots, n\}$ with $i+j$ cycles, there are $\binom{i+j}{i}$ ways to distribute them into C_1 and C_2 . Since we can reconstruct τ and σ from C_1 and C_2 , the number of $\pi = \Phi(T)$ is equal to $c(n-1, i+j) \binom{i+j}{i}$. \square

3. UNRESTRICTED COLUMNS AND SIGN-IMBALANCE

Recall that $\text{urc}(T)$ denotes the number of unrestricted columns of T . We define

$$P_t(x) = \sum_{n \geq 0} \sum_{T \in \mathcal{PT}(n)} t^{\text{urc}(T)} x^n.$$

Then Theorem 1.1 is the same as the following.

Theorem 3.1. *We have*

$$P_t(x) = \frac{1 + E_t(x)}{1 + (t-1)x E_t(x)},$$

where

$$E_t(x) = \sum_{n \geq 1} n(t)_{n-1} x^n.$$

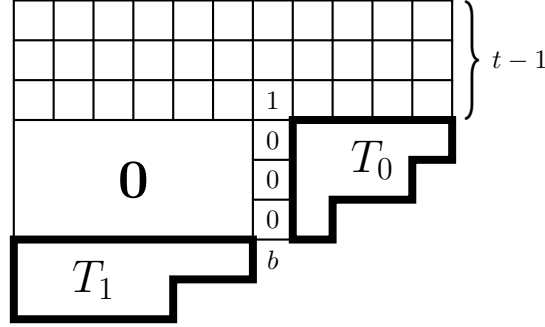
Proof. We will prove the following equivalent formula:

$$(4) \quad 1 + E_t(x) = P_t(x) + (t-1)x E_t(x) P_t(x).$$

Since the coefficients of x^n in both sides are polynomials in t , it is sufficient to prove (4) for all positive integers t . Assume that t is a positive integer.

We use a similar argument as in the proof of Theorem 2.2. However, we will use the permutation tableau itself instead of the alternative representation.

Given $T \in \mathcal{PT}(n)$, let T' be the permutation tableau in $\mathcal{PT}(n+t-1)$ obtained from T by adding $t-1$ south steps at the beginning, where the first $t-1$ rows (the added rows) have 0's only. Now consider all permutation tableaux obtained from T' by replacing the first $t-1$ 0's with i 0's and $t-1-i$ 1's for some $0 \leq i \leq t-1$ in each column if it was an unrestricted column in T . Note that this replacement never violates the condition for a permutation tableau because we add 1's only on top of an unrestricted column. Clearly, there are $t^{\text{urc}(T)}$ such permutation tableaux for

FIGURE 4. The decomposition of Q .

each $T \in \mathcal{PT}(n)$. On the other hand, it is not difficult to see that these permutation tableaux are exactly those in $\mathcal{PT}(n+t-1)$ satisfying the following conditions.

- (1) Each integer $1 \leq i \leq t-1$ is the label of an unrestricted row.
- (2) The integer t is the label of a row.
- (3) When we remove the first $t-1$ rows, we get a permutation tableau.

Thus $P_t(x)$ is the generating function for the number of $P \in \mathcal{PT}(n+t-1)$ satisfying the above three conditions.

By Lemma 2.1, $P \in \mathcal{PT}(n+t-1)$ satisfies the conditions (1) and (2) if and only if $1, 2, \dots, t-1$ are RL-minima of π and t is an ascent of π for $\pi = \Phi(P) \in S_{n+t-1}$. Equivalently, $1, 2, \dots, t-1$ are arranged in this order and t is not immediately followed by any of $1, 2, \dots, t-1$ in π . The number of such π is equal to

$$(5) \quad \frac{(n+t-1)!}{(t-1)!} - (t-1) \cdot \frac{(n+t-2)!}{(t-1)!} = n \cdot t(t+1) \cdots (t+n-2) = n(t)_{n-1}.$$

Thus the left hand side of (4) is the generating function for the number of $P \in \mathcal{PT}(n+t-1)$ satisfying the conditions (1) and (2). Since $P_t(x)$ is the generating function for the number of $P \in \mathcal{PT}(n+t-1)$ satisfying the conditions (1), (2), and (3), it is enough to show that $(t-1)x E_t(x) P_t(x)$ is the generating function for the number of $P \in \mathcal{PT}(n+t-1)$ satisfying the conditions (1) and (2), but not (3).

Now consider $Q \in \mathcal{PT}(n+t-1)$ satisfying the conditions (1) and (2), but not (3). We will decompose Q as shown in Figure 4. Let Q' be the tableau obtained from Q by removing the first $t-1$ rows. Then Q' must have at least one zero column. Let b be the label of the leftmost zero column of Q' . Since Q is a permutation tableau, in Q , Row $t-1$ must have a 1 in Column $b+t-1$. Moreover, by the condition for a permutation tableau, all cells to the left of the cells in zero columns of Q' have 0's only. Then in Figure 4, T_1 together with the part of the first $t-1$ rows of Q to the left of Column $b+t-1$ is a permutation tableau satisfying the conditions (1), (2), and (3), and T_0 together with the part of the first $t-1$ rows of Q to the right of Column $b+t-1$ is a permutation tableau satisfying the conditions (1) and (2). Note that the length of T_0 is at least 1 because the first step (a south step) of T_0 exists, and the length of T_1 may be 0. Since Column $b+t-1$ contributes the factor $(t-1)x$, we obtain that $(t-1)x E_t(x) P_t(x)$ is the desired generating function. \square

For the rest of this section we study the special cases $t = 2$ and $t = -1$ of Theorem 3.1.

3.1. The case $t = 2$: connected permutations. When $t = 2$ in Theorem 3.1, we get the following.

Corollary 3.2. *We have*

$$P_2(x) = \frac{1}{x} \left(1 - \frac{1}{\sum_{n \geq 0} n!x^n} \right).$$

Proof. Since

$$E_1(x) = \sum_{n \geq 1} n!x^n, \quad E_2(x) = \sum_{n \geq 1} n \cdot n!x^n = \frac{1-x}{x} E_1(x) - 1,$$

by Theorem 3.1, we get

$$\begin{aligned} P_2(x) &= \frac{1 + E_2(x)}{1 + x \cdot E_2(x)} = \frac{\frac{1-x}{x} E_1(x)}{1 + (1-x)E_1(x) - x} \\ &= \frac{1}{x} \cdot \frac{E_1(x)}{1 + E_1(x)} = \frac{1}{x} \left(1 - \frac{1}{1 + E_1(x)} \right) \\ &= \frac{1}{x} \left(1 - \frac{1}{\sum_{n \geq 0} n!x^n} \right). \end{aligned}$$

□

A *connected permutation* is a permutation $\pi = \pi_1 \cdots \pi_n \in S_n$ such that π does not have a proper prefix which is a permutation, in other words, there is no integer $k < n$ satisfying $\pi_1 \cdots \pi_k \in S_k$. Let $f(n)$ denote the number of connected permutations of $[n]$. It is known that $\sum_{n \geq 0} f(n)x^n = 1 - \left(\sum_{n \geq 0} n!x^n \right)^{-1}$, see [12]. Thus, by Corollary 3.2, we get the following.

Corollary 3.3. *We have*

$$\sum_{T \in \mathcal{PT}(n)} 2^{\text{urc}(T)} = f(n+1).$$

Now we prove Corollary 3.3 combinatorially. To do this we introduce the following terminology.

A *shift-connected permutation* is a permutation $\pi = \pi_1 \cdots \pi_n \in S_n$ satisfying the following: if $\pi_j = 1$, there is no integer $i < j$ such that $\pi_i \pi_{i+1} \cdots \pi_j \in S_{j-i+1}$. Note that if $\pi_n = 1$ then π is not a shift-connected permutation because $\pi_1 \cdots \pi_n \in S_n$.

Proposition 3.4. *The number of connected permutations of $[n]$ is equal to the number of shift-connected permutations of $[n]$.*

Proof. Let $\text{CP}(n)$ (resp. $\text{SCP}(n)$) denote the set of connected (resp. shift-connected) permutations of $[n]$. It is sufficient to find a bijection between $S_n \setminus \text{CP}(n)$ and $S_n \setminus \text{SCP}(n)$.

Given $\pi = \pi_1 \cdots \pi_n \in S_n \setminus \text{CP}(n)$, we define $\pi' \in S_n \setminus \text{SCP}(n)$ as follows. Since π is not a connected permutation, we can take the smallest integer $k < n$ such that $\sigma = \pi_1 \cdots \pi_k \in S_k$. We decompose π as $\pi = \sigma \tau(k+1) \rho$. Define $\pi' = \tau \sigma^+ 1 \rho$ where $\sigma^+ = \pi_k^+ \pi_{k-1}^+ \cdots \pi_1^+$ with $\pi_i^+ = \pi_i + 1$ for $i \in [k]$. It is easy to see that $\pi \mapsto \pi'$ is a bijection between $S_n \setminus \text{CP}(n)$ and $S_n \setminus \text{SCP}(n)$. □

The following proposition gives us a relation between permutation tableaux and shift-connected permutations.

Proposition 3.5. *Let $T \in \mathcal{PT}(n)$ and $\pi = \Phi(T)$. Then T has a column which has a 1 only in the first row if and only if π is not a shift-connected permutation.*

Proof. Assume that T has a column which has a 1 only in the first row. Then, in the alternative representation, T has a column, say Column d , which has a \uparrow in the first row and each cell except the topmost cell in the column has a \leftarrow in the cell or pointing to it. We will refer this condition as (*).

Recall the bijection Φ . To construct $\pi = \Phi(T)$ we first let π be the word consisting of the labels of the unrestricted rows of T arranged in increasing order. Then starting from the leftmost column we add some integers to π . Assume that we have proceeded just before Column d . We can decompose π as $\sigma 1 \tau$ at this stage. By the condition (*), neither σ nor τ has an integer i with $2 \leq i \leq d$. When we continue the construction of Φ , by the condition (*) again, no integer i with $2 \leq i \leq d$ is added before the last integer of σ or after 1. Thus at the end we must have $\Phi(T) = \pi = \sigma \rho 1 \tau$, which implies that $\rho 1 \in S_d$. Thus π is not a shift-connected permutation.

Conversely, assume that π is not a shift-connected permutation. Then we can decompose π as $\sigma \rho 1 \tau$ such that $\rho 1 \in S_d$ for some d . We claim that Column d in T has the condition (*). Since d is the largest integer in ρ , it is a RL-maximum in $\sigma \rho$. Thus Column d in T has a \uparrow in the first row by Lemma 2.1. If there is an unrestricted row with label $1 < i < d$, then i is a RL-minimum of τ contradicting to $\rho 1 \in S_d$. Hence, it remains to show that there is no row with label $1 < i < d$ having a \leftarrow in a column with label $d' > d$. Suppose there is such a row. We can assume that i is the smallest integer satisfying this condition. Let i' be the label of the row containing the unique \uparrow in Column d' . Since $1 \leq i' < i$, we must have $i' = 1$ because otherwise Row i' satisfies the above condition which is a contradiction to the minimality of i . Thus Column d' has a \uparrow in the first row. By Lemma 2.1, d' is a RL-maximum of $\sigma \tau$. Since Row i has a \leftarrow in Column d' , we must have i before d' in π . Since both d and d' are RL-maxima of $\sigma \tau$ and $d' > d$, we have d' before d . Thus the integers i, d', d appear in this order in π . This is a contradiction to $\rho 1 \in S_d$. \square

Using the argument in the proof of Theorem 3.1 one can easily see that $\sum_{T \in \mathcal{PT}(n)} 2^{\text{urc}(T)}$ is the number of $T \in \mathcal{PT}(n+1)$ without a column containing a 1 only in the first row. Thus, by Proposition 3.5, we get the following.

Proposition 3.6. *The number of shift-connected permutations of $[n+1]$ is equal to*

$$\sum_{T \in \mathcal{PT}(n)} 2^{\text{urc}(T)}.$$

Combining Propositions 3.4 and 3.6, we get a combinatorial proof of Corollary 3.3.

3.2. The case $t = -1$: sign-imbalance. When $t = -1$ in Theorem 3.1, we get the following.

Corollary 3.7. *We have*

$$P_{-1}(x) = \frac{1-x}{1-2x+2x^2}.$$

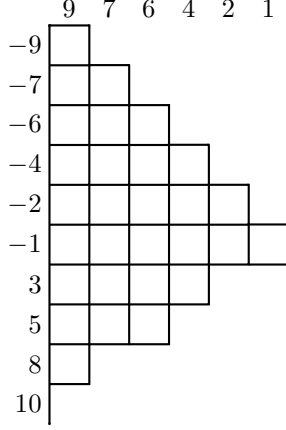


FIGURE 5. The shifted Ferrers diagram \overline{F} for the Ferrers diagram F in Figure 1.

Proof. Since $E_{-1}(x) = x - 2x^2$, we obtain

$$P_{-1}(x) = \frac{1 + E_{-1}(x)}{1 - 2x \cdot E_{-1}(x)} = \frac{1 + x - 2x^2}{1 - 2x^2 + 4x^3} = \frac{1 - x}{1 - 2x + 2x^2}.$$

□

Recall that $\text{sign}(T) = (-1)^{\text{urc}(T)}$ for $T \in \mathcal{PT}(n)$. Since

$$P_{-1}(x) = \frac{1 - x}{1 - 2x + 2x^2} = \frac{1}{2} \cdot \left(\frac{1}{1 - (1+i)x} + \frac{1}{1 - (1-i)x} \right),$$

we get the following sign-imbalance formula for permutation tableaux.

Corollary 3.8. *We have*

$$(6) \quad \sum_{T \in \mathcal{PT}(n)} \text{sign}(T) = \frac{(1+i)^n + (1-i)^n}{2}.$$

Equivalently, if $n = 4k + r$ for $0 \leq r < 4$, then

$$\sum_{T \in \mathcal{PT}(n)} \text{sign}(T) = \begin{cases} (-1)^k \cdot 2^{2k}, & \text{if } r = 0 \text{ or } r = 1, \\ 0, & \text{if } r = 2, \\ (-1)^{k+1} \cdot 2^{2k+1}, & \text{if } r = 3. \end{cases}$$

4. TYPE B PERMUTATION TABLEAUX

For a Ferrers diagram F with k columns, the *shifted* Ferrers diagram of F , denoted \overline{F} , is the diagram obtained from F by adding k rows of size $1, 2, \dots, k$ above it in increasing order, see Figure 5. The *length* of \overline{F} is defined to be the length of F . The *diagonal* of \overline{F} is the set of rightmost cells in the added rows. A *diagonal cell* is a cell in the diagonal. We label the added rows as follows. If the diagonal cell of an added row is in Column i , then the row is labeled with $-i$.

A *type B permutation tableau* of length n is a shifted Ferrers diagram \overline{F} of length n together with a 0, 1-filling of \overline{F} satisfying the following conditions (see Figure 6):

- (1) each column has at least one 1,

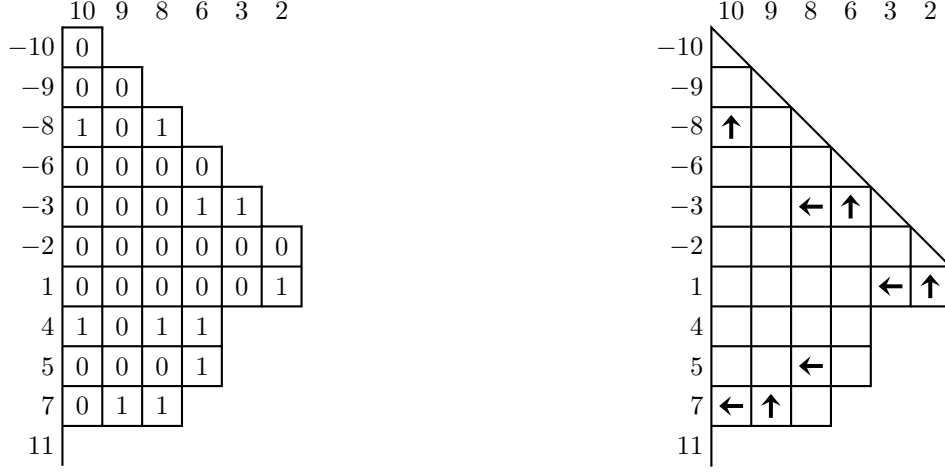


FIGURE 6. A type B permutation tableau T (left) and the alternative representation of T (right).

- (2) there is no 0 which has a 1 above it in the same column and a 1 to the left of it in the same row,
- (3) if a 0 is in a diagonal cell, then there is no 1 to the left of it in the same row.

Note that the condition (3) above means that if we add the copy of the tableau below the diagonal cells reflected about the diagonal line to get a symmetric tableau, the condition (2) still holds.

We denote by $\mathcal{PT}_B(n)$ the set of type B permutation tableaux of length n . For $T \in \mathcal{PT}_B(n)$, we say that a 0 is *restricted* if it has a 1 above it in the same column or it is in a diagonal cell. If a row of T does not contain any restricted 0, it is called an *unrestricted row*.

The *alternative representation* of $T \in \mathcal{PT}_B(n)$ is the diagram obtained from T as follows. Firstly, we replace the topmost 1's with \uparrow 's and the rightmost restricted 0's with \leftarrow 's and remove the remaining 0's and 1's. Secondly, we remove the \uparrow 's in the diagonal and cut off the diagonal cells as shown in Figure 6. Note that in the alternative representation of $T \in \mathcal{PT}_B(n)$, there is no arrow pointing to another arrow. Here we consider the cutting segment as a mirror, that is, an arrow \uparrow in Column m is pointing to not only the cells above it in Column m but also the cells in Row $-m$. Conversely, for any diagram D satisfying this condition, we have $T \in \mathcal{PT}_B(n)$ whose alternative representation is the diagram D . In contrast to the alternative representation of a permutation tableau, we may have a column without a \uparrow , which is in fact hidden in a diagonal cell. We note that the alternative representation of $T \in \mathcal{PT}_B(n)$ is in fact a half of a symmetric alternative tableau in [9, Section 3.5].

From now on, we will only consider the alternative representation unless otherwise stated. Note that Row i is unrestricted if and only if Row i has no \leftarrow and Column $|i|$, if it exists, has no \uparrow .

A *type B permutation* of $[n]$ is a permutation of $[n]$ in which each integer may be negated. For example, $4, -1, 2, 5, -3$ is a type B permutation of $[5]$. We denote by B_n the set of type B permutations of $[n]$.

Now we define a map $\Phi_B : \mathcal{PT}_B(n) \rightarrow B_n$. Let $T \in \mathcal{PT}_B(n)$. Suppose T has k columns labeled with $c_1 < c_2 < \dots < c_k$. We set $\pi^{(k+1)}$ to be the word of labels of the unrestricted rows of T arranged in increasing order. For i from k to 1 , we define $\pi^{(i)}$ to be the word obtained from $\pi^{(i+1)}$ by adding some integers as follows. If Column c_i has a \uparrow , let r_0 be the label of the row containing the \uparrow and add c_i before r_0 in $\pi^{(i+1)}$. If Column c_i has no \uparrow , then do nothing (in this case $-c_i$ should be already in $\pi^{(i+1)}$). If $r_1 < \dots < r_s$ are the labels of the rows containing a \leftarrow in Column c_i , we add r_1, \dots, r_s in this (increasing) order before c_i or $-c_i$ in π . We define $\Phi_B(T) = \pi^{(1)}$. For example, if T is the permutation tableau in Figure 6, then

$$(7) \quad \Phi_B(T) = 9, 7, 10, 6, 2, 1, -3, 5, -8, 4, 11.$$

The map $\Phi_B : \mathcal{PT}_B(n) \rightarrow B_n$ is a generalization of $\Phi : \mathcal{PT}(n) \rightarrow S_n$ in the sense that if $T \in \mathcal{PT}_B(n)$ has no diagonal 1, then we can consider T as an element in $\mathcal{PT}(n)$ and the map Φ_B on such T is identical with Φ .

Proposition 4.1. *The map $\Phi_B : \mathcal{PT}_B(n) \rightarrow B_n$ is a bijection.*

We will prove Proposition 4.1 by constructing the inverse of Φ_B . In order to do this we need the following definitions. For $\pi = \pi_1 \pi_2 \dots \pi_n \in B_n$ and $i \in [n]$, we say that π_i is a *descent* of π if $\pi_i > \pi_{i+1}$, and an *ascent* otherwise, where $\pi_{n+1} = n+1$. Note that our definition is different from the usual one which uses the index i as a descent or an ascent instead of the value π_i . We also say that π_i is a *signed descent* of π if $\pi_i < 0$ or $\pi_i > |\pi_{i+1}|$, and a *signed ascent* otherwise. The *shape* of π is the shifted Ferrers diagram such that $|\pi_i|$ is the label of a column if and only if π_i is a signed descent of π . For example, the shape of the type B permutation in (7) is the same as the shape of the type B permutation tableau in Figure 6.

Now we define a map $\Psi_B : B_n \rightarrow \mathcal{PT}_B(n)$. Given $\pi = \pi_1 \pi_2 \dots \pi_n \in B_n$, we first set T to be the empty tableau with the same shape as π . Let $\pi_{d_1}, \pi_{d_2}, \dots, \pi_{d_k}$ be the signed descents of π with $|\pi_{d_1}| < |\pi_{d_2}| < \dots < |\pi_{d_k}|$. Let $\pi^{(1)} = \pi$. For i from 1 to k , we define $\pi^{(i+1)}$ using $\pi^{(i)}$ as follows. During the process, we will fill the cells in Column $|\pi_{d_i}|$ of T . Suppose $\pi^{(i)} = \sigma_1 \sigma_2 \dots \sigma_\ell$ and $\pi_{d_i} = \sigma_d$.

Case 1: If $\sigma_d < 0$, let r be the smallest integer with $\sigma_d < \sigma_r < \sigma_{r+1} < \dots < \sigma_{d-1} < |\sigma_d|$. We fill the $(\sigma_j, |\sigma_d|)$ -entry with a \leftarrow for each $r \leq j \leq d-1$. Then $\pi^{(i+1)}$ is the word obtained from $\pi^{(i)}$ by removing σ_j for all $r \leq j \leq d-1$.

Case 2: If $\sigma_d > 0$, then we have $\sigma_d > |\sigma_{d+1}|$ because $\pi_{d_i} = \sigma_d$ is a signed descent of $\pi^{(i)}$ (this will be proved in Lemma 4.2). Let r be the smallest integer with $\sigma_{d+1} < \sigma_r < \sigma_{r+1} < \dots < \sigma_{d-1} < \sigma_d$. We fill the (σ_{d+1}, σ_d) -entry with a \uparrow and the (σ_j, σ_d) -entry with a \leftarrow for each $r \leq j \leq d-1$. Then $\pi^{(i+1)}$ is the word obtained from $\pi^{(i)}$ by removing σ_j for all $r \leq j \leq d$.

Repeat this process until we get $\pi^{(k+1)}$ and define $\Psi_B(\pi)$ to be the resulting tableau T . For example, if π is the permutation in (7), then $\Psi_B(\pi)$ is the permutation tableau in Figure 6. From the construction of Ψ_B it is easy to see that $\pi^{(k+1)}$ consists of the integers which are the labels of the unrestricted rows of $\Psi_B(\pi)$.

Lemma 4.2. *Following the notations in the definition of Ψ_B , we have the following.*

- (1) For each $i \in [k]$, π_{d_i} is a signed descent in $\pi^{(i)}$.
- (2) $\pi^{(k+1)}$ is the word of the RL-minima of π arranged in increasing order.

Proof. (1) It is enough to show the following: for $j < i$, when we go from $\pi^{(j)}$ to $\pi^{(j+1)}$, π_{d_i} is not deleted and it remains a signed descent. First observe that we have $|\pi_{d_j}| < |\pi_{d_i}|$ by definition. Since we only delete integers between $-|\pi_{d_j}|$ and $|\pi_{d_j}|$, π_{d_i} is not deleted. If $\pi_{d_i} < 0$, it is a signed descent. Otherwise, the integer to the right of π_{d_i} may change. In this case, however, the integer to the right of π_{d_i} in $\pi^{(j+1)}$ is smaller than $|\pi_{d_j}|$ by the construction of Ψ_B . Thus π_{d_i} remains a signed descent in $\pi^{(j+1)}$.

(2) Similarly, we can prove the following.

- For an integer g in $\pi^{(i)}$, g is a descent in $\pi^{(i)}$ if and only if g is a descent in π .
- $\pi^{(k+1)}$ does not have any descent of π .
- $\pi^{(k+1)}$ contains all RL-minima of π .

Using the above three statements one can easily derive (2). \square

By Lemma 4.2, $\pi^{(k+1)}$ is the RL-minima of π arranged in increasing order and these integers are the labels of the unrestricted rows of $T = \Psi_B(\pi)$. Now it is easy to see that $\Phi_B(T) = \pi$. Hence we obtain that the map $\Phi_B : \mathcal{PT}_B(n) \rightarrow B_n$ is a bijection with inverse $\Phi_B^{-1} = \Psi_B$. This completes the proof of Proposition 4.1.

Now we will find some properties of Φ_B analogous to Φ .

Let $T \in \mathcal{PT}_B(n)$ and let Row m be the topmost unrestricted row of T . We denote by $\text{diag}(T)$ the number of columns without a \uparrow , by $\text{urr}(T)$ the number of unrestricted rows and by $\text{top}_{0,1}(T)$ the number of arrows in Row m and Column $|m|$. Equivalently, in the original permutation tableau representation $\text{diag}(T)$ is the number of diagonal cells containing a 1 and $\text{top}_{0,1}(T)$ is the number of 1's in Row m except the 1 in the diagonal cell, if it exists, plus the number of rightmost restricted 0's in Column $|m|$. Note that if $m > 0$, then we must have $m = 1$ and $\text{top}_{0,1}(T)$ is simply the number of 1's in Row 1.

Example 4.1. Let T be the type B permutation tableau in Figure 6. Then the topmost unrestricted row of T is Row -8 and $\text{diag}(T) = 2$, $\text{urr}(T) = 3$ and $\text{top}_{0,1}(T) = 1 + 2 = 3$.

The following is a type B analog of Lemma 2.1.

Proposition 4.3. Let $T \in \mathcal{PT}_B(n)$ and $\pi = \Phi_B(T)$ with π decomposed as $\pi = \sigma\tau m\rho$, where $\min(\pi) = m$, the last element of σ is greater than $|m|$ and each element of τ is smaller than $|m|$. Then we have the following.

- (1) Row m is the topmost unrestricted row of T .
- (2) Column t has no \uparrow if and only if $-t$ appears in π .
- (3) The labels of the unrestricted rows of T are exactly the RL-minima of π .
- (4) The labels of the columns of T with a \uparrow in Row m are exactly the RL-maxima of σ . Here σ can be empty.
- (5) The labels of the rows of T containing a \leftarrow in Column $|m|$ are exactly the RL-minima of τ . Here τ can be empty.
- (6) We have $\text{neg}(\pi) = \text{diag}(T)$.

Proof. Let T have k columns labeled with $c_1 < c_2 < \dots < c_k$. Using the notations in the definition of Φ_B , we can observe the following.

Observation: When we obtain $\pi^{(i)}$ from $\pi^{(i+1)}$, we add an increasing sequence of integers $a_1 < \dots < a_s$ before b for some integers a_1, \dots, a_s, b satisfying $b < a_s \leq c_i$.

Then (2) is obvious from the definition of Φ_B and (3) follows from the fact that we do not create any new RL-minimum when we obtain $\pi^{(i)}$ from $\pi^{(i+1)}$ by the above observation. Since the label of the topmost unrestricted row is smaller than that of any other unrestricted row, by (3) we obtain (1).

Now we prove (4) and (5). Let t be the integer with $c_t = |m|$ if $m < 0$ and $t = k + 1$ if $m = 1$. Since $\pi^{(k+1)}$ is the word of the RL-minima of π arranged in increasing order, the first element of $\pi^{(k+1)}$ is m . Thus for each $i \in [k + 1]$ we can decompose $\pi^{(i)}$ as $\pi^{(i)} = \sigma^{(i)} \tau^{(i)} m \rho^{(i)}$, where the last element of $\sigma^{(i)}$ is greater than $|m|$ and each element of $\tau^{(i)}$ is smaller than $|m|$.

Using the observation, it is not difficult to see that $\text{RLmax}(\sigma^{(i)}) = \text{RLmax}(\sigma^{(i+1)}) \cup \{c_i\}$ if Column $|c_i|$ has a \uparrow in Row m , and $\text{RLmax}(\sigma^{(i)}) = \text{RLmax}(\sigma^{(i+1)})$ otherwise. Thus we get (4). Using the observation again, one can also easily see that $\tau^{(k+1)} = \tau^{(k)} = \dots = \tau^{(t-1)} = \emptyset$, $\tau^{(t)}$ is the increasing sequence of labels of the rows of T containing a \leftarrow in Column $|m|$ and $\text{RLmin}(\tau^{(t)}) = \text{RLmin}(\tau^{(t-1)}) = \dots = \text{RLmin}(\tau^{(1)})$. Thus we get (5).

From the construction of Φ_B , (6) is obvious. \square

Now we are ready to prove the following theorem.

Theorem 4.4. *We have*

$$\sum_{T \in \mathcal{PT}_B(n)} x^{\text{urr}(T)-1} y^{\text{top}_{0,1}(T)} z^{\text{diag}(T)} = (1+z)^n (x+y)_{n-1}.$$

Note, if $z = 0$, the above theorem reduces to Theorem 2.2 by identifying a permutation tableau with a type B permutation tableau without any diagonal 1. We give two proofs of this theorem. These are type B analogs of the two proofs of Theorem 2.2.

First proof of Theorem 4.4. We consider x and y as positive integers. Let $N = n + x + y - 2$. Note $T \in \mathcal{PT}_B(n)$ with $\text{diag}(T) = 0$ can be considered as an element in $\mathcal{PT}(n)$. Thus by Theorem 2.2, the left hand side of the formula is equal to

$$(8) \quad (x+y)_{n-1} + \sum_{\substack{T \in \mathcal{PT}_B(n) \\ \text{diag}(T) \neq 0}} x^{\text{urr}(T)-1} y^{\text{top}_{0,1}(T)} z^{\text{diag}(T)}.$$

Given $T \in \mathcal{PT}_B(n)$ with $\text{diag}(T) \neq 0$, we define $T' \in \mathcal{PT}_B(N)$ as follows. Let Row m be the topmost unrestricted row of T . Since $\text{diag}(T) \neq 0$ we have $m < 0$. Then T' is obtained from T by inserting $y - 1$ columns to the left of Column $|m|$, $y - 1$ rows above Row m and $x - 1$ columns at the end, where the new $x - 1$ cells attached to the left of Row m are fill with \uparrow 's as shown Figure 7.

As we did in the first proof of Theorem 2.2, we can move each arrow in Row m (resp. Column $|m|$) to one of the $y - 1$ cells above it (resp. to the left of it), and we can place a \leftarrow in one of the $x - 1$ cells to the left of each unrestricted row, except Row m . Let $X(T)$ denote the set of type B permutation tableaux obtained in this way. Clearly $X(T)$ has $x^{\text{urr}(T)-1} y^{\text{top}_{0,1}(T)}$ elements and for each $Q \in X(T)$ we have

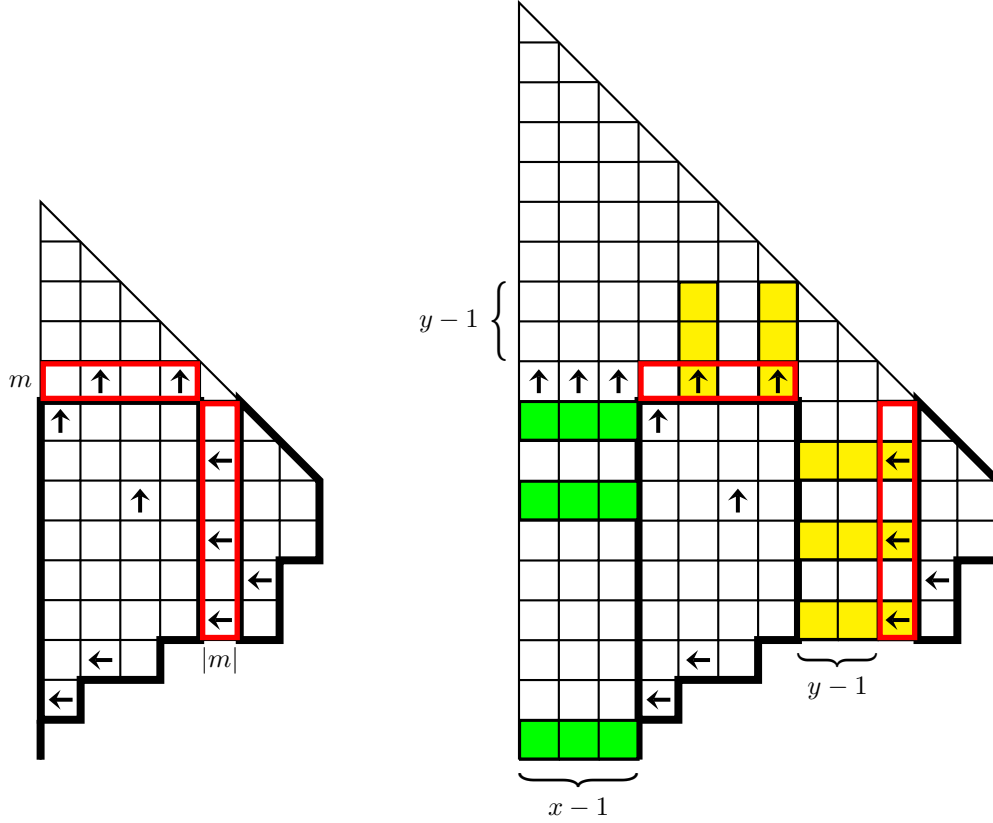


FIGURE 7. Expansion

$z^{\text{diag}(Q)} = z^{\text{diag}(T)+y-1}$. Thus (8) is equal to

$$(9) \quad (x+y)_{n-1} + \sum_{\substack{T \in \mathcal{PT}_B(n) \\ \text{diag}(T) \neq 0}} \sum_{Q \in X(T)} z^{\text{diag}(Q)-y+1}.$$

On the other hand, one can check that $Q \in X(T)$ for some $T \in \mathcal{PT}_B(n)$ with $\text{diag}(T) \neq 0$ if and only if Q satisfies the following conditions. Let Row M be the topmost unrestricted row of Q .

- We have $M \leq -y$ and $|M| \leq N+x-1$.
- Row r is unrestricted for all r with $M \leq r \leq M+y-1$.
- Column c has a \uparrow in Row M for all c with $N-x+2 \leq c \leq N$.

Now let $\pi = \Phi_B(Q) \in B_N$. By Proposition 4.3, Q satisfies the above conditions if and only if $M = \min(\pi) \leq -y$, $|M| \leq N+x-1$ and the integers $N, N-1, \dots, N-x+2, M, M+1, \dots, M+y-1$ are arranged in this order in π . Suppose $\pi \in B_n$ satisfies these conditions. Let $A \subset [N]$ be the set of integers a such that π has $-a$. Since $N, N-1, \dots, N-x+2$ appear in π , we must have $A \subset [n+y-1]$. Since $M \leq -y$ and $M, M+1, \dots, M+y-1$ appear in π , the largest y elements in A are consecutive. Let A' be the set obtained from A by deleting the largest $y-1$ elements. Then $\emptyset \neq A' \subset [n]$. Note that we can reconstruct A from A' . When A' is fixed, the

number of such $\pi \in B_N$ is equal to $\frac{N!}{(x+y-1)!} = (x+y)_{n-1}$. Since $\text{diag}(Q) = \text{neg}(\pi)$, the number of negative integers in π , and $\text{neg}(\pi) = |A| = |A'| + y - 1$, we obtain that (9) is equal to the following which finishes the proof:

$$(x+y)_{n-1} + \sum_{\emptyset \neq A' \subset [n]} z^{|A'|} (x+y)_{n-1} = (1+z)^n (x+y)_{n-1}.$$

□

Second proof of Theorem 4.4. Let $T \in \mathcal{PT}_B(n)$ and $\pi = \Phi_B(T)$. We decompose π as $\pi = \sigma\tau m\rho$, where $m = \min(\pi)$, the last element of σ is greater than $|m|$ and all the entries in τ are smaller than $|m|$. Then by Proposition 4.3, $\text{diag}(T) = \text{neg}(\pi)$, $\text{urr}(T) - 1 = \text{RLmin}(\rho)$, $\text{top}_{0,1}(T) = \text{RLmax}(\sigma) + \text{RLmin}(\tau)$, where for a word w of integers, $\text{RLmin}(w)$ (resp. $\text{RLmax}(w)$) denotes the number of RL-minima (resp. RL-maxima) of w . Thus

$$(10) \quad \sum_{T \in \mathcal{PT}_B(n)} x^{\text{urr}(T)-1} y^{\text{top}_{0,1}(T)} z^{\text{diag}(T)} = \sum_{\pi \in B_n} z^{\text{neg}(\pi)} x^{\text{RLmin}(\rho)} y^{\text{RLmax}(\sigma) + \text{RLmin}(\tau)}.$$

On the other hand, consider the set C_1 (resp. C_2 and C_3) of cycles of σ (resp. τ and ρ) according to RL-maxima (resp. RL-minima and RL-minima). Then

$$z^{\text{neg}(\pi)} x^{\text{RLmin}(\rho)} y^{\text{RLmax}(\sigma) + \text{RLmin}(\tau)} = z^{\text{neg}(\pi)} x^{|C_3|} y^{|C_1| + |C_2|}.$$

Note that π is determined by the ordered triple (C_1, C_2, C_3) and m . Moreover, since each cycle in C_1 contains at least one integer greater than $|m|$ and no cycle in C_2 has this property, π is determined by the ordered pair $(C_1 \cup C_2, C_3)$ and m . Let us fix the set $A \subset [n]$ of integers a such that π has $-a$. Note that A determines the set S of integers in π and $m = \min(S)$. Since $(C_1 \cup C_2) \cup C_3$ is the set of cycles of a permutation on $S \setminus \{m\}$, the right hand side of (10) is equal to

$$\sum_{A \subset [n]} z^{|A|} \sum_{i,j} c(n-1, i+j) \binom{i+j}{i} x^i y^j,$$

which is equal to $(1+z)^n (x+y)_{n-1}$ as we have shown in the second proof of Theorem 2.2. □

5. ZIGZAG MAPS

Steingrímsson and Williams [15] defined the *zigzag map* $\zeta^{\text{perm}} : \mathcal{PT}(n) \rightarrow S_n$ on permutation tableaux as follows. A *zigzag path* on a permutation tableau $T \in \mathcal{PT}(n)$ is a path entering from the left of a row or the top of a column, going to the east or to the south changing the direction alternatively whenever it meets a 1 until exiting the tableau, see Figure 8. Then $\zeta^{\text{perm}}(T)$ is defined to be the permutation $\pi = \pi_1 \cdots \pi_n$ where $\pi_i = j$ if the zigzag path starting from Row i or Column i exits T from Row j or Column j . It is easy to see that $\zeta^{\text{perm}}(T)$ is a permutation of $[n]$.

Steingrímsson and Williams [15] showed that the map $\zeta^{\text{perm}} : \mathcal{PT}(n) \rightarrow S_n$ is in fact a bijection preserving many interesting statistics. Similarly, we can define the zigzag map $\zeta^{\text{alt}} : \mathcal{PT}(n) \rightarrow S_n$ on the alternative representation in the same way: whenever we meet an arrow we change the direction, see Figure 8.

In this section we show that ζ^{alt} is essentially the same as the first bijection $\Phi : \mathcal{PT}(n) \rightarrow S_n$ of Corteel and Nadeau [3].

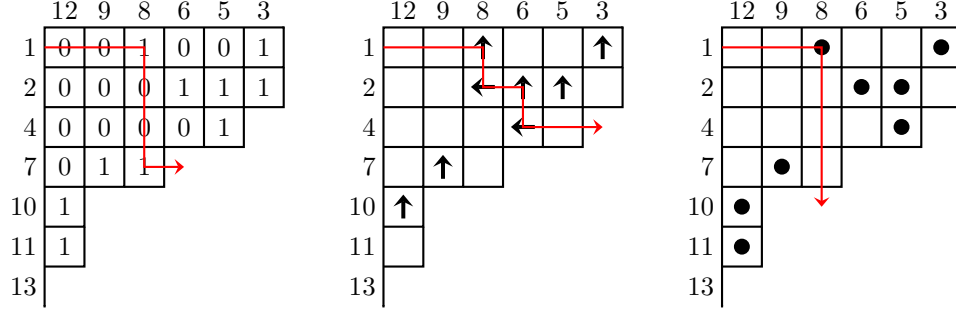


FIGURE 8. Zigzag paths on a permutation tableau T (left), on the alternative representation of T (middle) and on the bare representation of T (right).

Given $T \in \mathcal{PT}(n)$, we can consider $\zeta^{\text{alt}}(T)$ as a product of cycles as follows. Assume that T has k columns with labels $c_1 < c_2 < \dots < c_k$. Note that each column has exactly one \uparrow and may have several \leftarrow 's. For each Column c_i let C_i be the cycle $(r_1^{(i)}, r_2^{(i)}, \dots, r_{\ell_i}^{(i)}, c_i, r_0^{(i)})$ where $r_0^{(i)}$ is the label of the row containing the unique \uparrow in Column c_i and $r_1^{(i)} < r_2^{(i)} < \dots < r_{\ell_i}^{(i)}$ are the labels of the rows containing a \leftarrow in Column c_i . It is easy to see that $\zeta^{\text{alt}}(T)$ is equal to the cycle product $C_1 C_2 \dots C_k$. For example, if T is the permutation tableau in Figure 8, then

$$\zeta^{\text{alt}}(T) = (3, 1)(5, 2)(4, 6, 2)(2, 8, 1)(9, 7)(12, 10).$$

Let $\pi^{(i)} = C_i C_{i+1} \dots C_k$ for $i \in [k+1]$. Thus $\pi^{(1)}$ is equal to $\zeta^{\text{alt}}(T)$ and $\pi^{(k+1)}$ is the identity. We represent $\pi^{(i)}$ as a product of cycles as follows. First we represent $\pi^{(k+1)} = (m_1)(m_2) \dots (m_s)$ where $m_1 < m_2 < \dots < m_s$ are the labels of the unrestricted rows of T . For each $i \in [k]$, we represent $\pi^{(i)}$ using the representation of $\pi^{(i+1)}$ as follows. From the construction it is easy to see that among the elements in the cycle $C_i = (r_1^{(i)}, r_2^{(i)}, \dots, r_{\ell_i}^{(i)}, c_i, r_0^{(i)})$, $r_0^{(i)}$ is the only integer which appears in the cycle product representation of $\pi^{(i+1)}$. Note that for different integers $a_1, \dots, a_p, b_1, \dots, b_q, d_1, \dots, d_r$ and x , we have

$$(a_1, \dots, a_p, x)(b_1, \dots, b_q, x, d_1, \dots, d_r) = (b_1, \dots, b_q, a_1, \dots, a_p, x, d_1, \dots, d_r).$$

By this product rule, we represent $\pi^{(i)}$ as the one obtained by inserting $r_1^{(i)}, r_2^{(i)}, \dots, r_{\ell_i}^{(i)}, c_i$ before $r_0^{(i)}$ in the representation of $\pi^{(i+1)}$. Since this insertion process is exactly the same as that in the definition of the map Φ , we obtain that $\pi^{(1)}$ is equal to $\varphi \circ \Phi(T)$. Here, for a permutation σ , $\varphi(\sigma)$ denotes the permutation consisting of the cycles of the word σ according to RL-minima. Thus we get the following theorem.

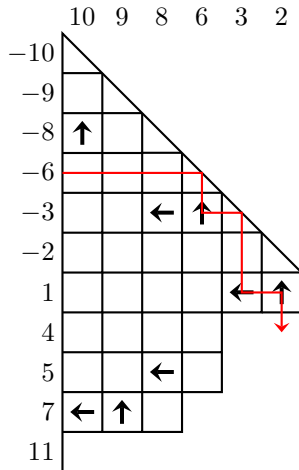
Theorem 5.1. *The zigzag map ζ^{alt} is the same as $\varphi \circ \Phi$.*

For example, if T is the permutation tableau in Figure 8, we have

$$\Phi(T) = 4, 6, 5, 2, 8, 3, 1, 9, 7, 12, 10, 11, 13,$$

$$\varphi \circ \Phi(T) = (4, 6, 5, 2, 8, 3, 1)(9, 7)(12, 10)(11)(13) = \zeta^{\text{alt}}(T).$$

Remark 5.1. Burstein [1] defined the *bare representation* of $T \in \mathcal{PT}(n)$ to be the diagram obtained from T by replacing the topmost or the leftmost 1's with



dots and removing the rest of the 0's and 1's, see Figure 8. In the same paper, he defined a map, say Θ , on the bare representation by considering it as a binary forest. Similarly, one can check that $\varphi \circ \Theta$ is in fact the zigzag map on the bare representation. As Nadeau mentioned in [9], though not explicitly stated, one can obtain a bare representation from an alternative representation (of another permutation tableau) very easily: replace all the arrows with dots. Since there is exactly one \uparrow in each column and it is the topmost arrow, we can reconstruct the alternative representation. It is worth noting that although the three bijections ζ^{perm} , Φ and Θ have very different descriptions, they are all zigzag maps on different representations.

For $T \in \mathcal{PT}_B(n)$ and $i \in [n]$, we define $i_T = i$ if Column i has a \uparrow , and $i_T = -i$ otherwise. Let $R(T) = \{1_T, 2_T, \dots, n_T\}$. Note that $R(T)$ is the set of integers appearing in the permutation $\Phi_B(T)$. Then we define $\zeta_B^{\text{alt}}(T)$ to be the map $\pi : R(T) \rightarrow R(T)$ such that for each $i \in [n]$, if the zigzag path entering from Row i or Row $-i$ exits T from Row j or Column j , then $\pi(i_T) = j_T$. It is easy to see that $\pi : R(T) \rightarrow R(T)$ is a bijection.

$$(11) \quad \pi = \begin{pmatrix} 1 & 2 & -3 & 4 & 5 & 6 & 7 & -8 & 9 & 10 & 11 \\ -3 & 1 & 5 & 4 & -8 & 2 & 10 & 9 & 7 & 6 & 11 \end{pmatrix},$$

where the bi-letter $\begin{smallmatrix} i \\ j \end{smallmatrix}$ means that $\pi(i) = j$.

Note that we can identify $\zeta_B^{\text{alt}}(T) = \pi$ with an element in B_n by reading the lower line in the two line notation of π as shown in (11). For example, the π in (11) is identified with $-3, 1, 5, 4, -8, 2, 10, 9, 7, 6, 11 \in B_{11}$. Thus ζ_B^{alt} is a map from $\mathcal{PT}_B(n)$ to B_n .

We extend the definition of the map φ on S_n to B_n as follows. For $\pi \in B_n$, we define $\varphi(\pi)$ to be the permutation on the set of integers in π consisting of the cycles of the word π according to RL-minima. For example, if $\sigma = 9, 7, 10, 6, 2, 1, -3, 5, -8, 4, 11$, then

$$\varphi(\sigma) = (9, 7, 10, 6, 2, 1, -3, 5, -8)(4)(11).$$

By the same argument used for proving Theorem 5.1, we obtain the following theorem.

Theorem 5.2. *The zigzag map $\zeta_B^{\text{alt}} : \mathcal{PT}_B(n) \rightarrow B_n$ is the same as $\varphi \circ \Phi_B$.*

For example, if T is the tableau in Figure 9, then

$$\begin{aligned} \Phi_B(T) &= 9, 7, 10, 6, 2, 1, -3, 5, -8, 4, 11, \\ \zeta_B^{\text{alt}}(T) &= \varphi \circ \Phi_B(T) = (9, 7, 10, 6, 2, 1, -3, 5, -8)(4)(11). \end{aligned}$$

6. FURTHER STUDY

In Corollary 3.8 we have found the following sign-imbalance formula for permutation tableaux: if $n = 4k + r$ for $0 \leq r < 4$, then

$$(12) \quad \sum_{T \in \mathcal{PT}(n)} \text{sign}(T) = \frac{(1+i)^n + (1-i)^n}{2} = \begin{cases} (-1)^k \cdot 2^{2k}, & \text{if } r = 0 \text{ or } r = 1, \\ 0, & \text{if } r = 2, \\ (-1)^{k+1} \cdot 2^{2k+1}, & \text{if } r = 3. \end{cases}$$

Since we have obtained (12) by putting $t = -1$ in the generating function in Theorem 3.1, we propose the following problem.

Problem 6.1. Find a combinatorial proof of (12).

There is also a sign-imbalance formula for standard Young tableaux. The sign of a standard Young tableau T is defined to be the sign of the permutation obtained by reading T like a book. For example,

$$\text{sign} \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \hline \end{array} \right) = \text{sign}(12534) = 1.$$

Stanley [14] conjectured the following sign-imbalance formula for standard Young tableaux:

$$(13) \quad \sum_{T \in \mathcal{SYT}(n)} \text{sign}(T) = 2^{\lfloor \frac{n}{2} \rfloor},$$

where $\mathcal{SYT}(n)$ is the set of standard Young tableaux with n squares. Lam [7] and Sjöstrand [11] independently proved (13), and Kim [6, Corollary 4.10 and Theorem 4.13] generalized (13) to skew standard Young tableaux.

If we take the absolute values of (12) and (13), we obtain the following somewhat unexpected result: if $n \not\equiv 2 \pmod{4}$, then

$$(14) \quad \left| \sum_{T \in \mathcal{PT}(n)} \text{sign}(T) \right| = \left| \sum_{T \in \mathcal{SYT}(n)} \text{sign}(T) \right| = 2^{\lfloor \frac{n}{2} \rfloor}.$$

In (14) taking the absolute value of the sign-imbalance for standard Young tableaux is only for an aesthetic reason. It will be interesting to find a combinatorial explanation for (14).

Since we do not have a type B analog for (12), we propose the following problem.

Problem 6.2. Define the sign of a type B permutation tableau and find a sign-imbalance for type B permutation tableaux.

ACKNOWLEDGEMENT

The authors would like to thank Matthieu Josuat-Vergès for helpful discussion. The authors also thank the anonymous referees for their careful reading and helpful comments, especially the comment for simplifying the proof of Theorem 3.1.

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